A perturbed nonlinear elliptic PDE with two Hardy-Sobolev critical exponents*

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Abstract

Let Ω be a C^1 open bounded domain in $\mathbb{R}^N$ ($N \geq 3$) with $0 \in \partial\Omega$. Suppose that $\partial\Omega$ is C^2 at 0 and the mean curvature of $\partial\Omega$ at 0 is negative. Consider the following perturbed PDE involving two Hardy-Sobolev critical exponents:

$$\begin{cases} \Delta u + \lambda_1 \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \lambda_2 \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} + \lambda_3 \frac{u^p}{|x|^{s_3}} = 0 \quad \text{in } \Omega, \\ u(x) > 0 \text{ in } \Omega, \quad u(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

where $0 < s_2 < s_1 < 2, 0 \leq s_3 < 2, 2^*(s_i) := \frac{2(N-s_i)}{N-2}, 0 \neq \lambda_i \in \mathbb{R}, \lambda_2 > 0, 1 < p \leq 2^*(s_3) - 1$. The existence of ground state solution is studied under different assumptions via the concentration compactness principle and the Nehari manifold method. We also apply a perturbation method to study the existence of positive solution.

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1 Introduction

Consider the existence of ground state solution to the following problem

$$\begin{cases} \Delta u + \lambda_1 \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \lambda_2 \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} + \lambda_3 \frac{u^p}{|x|^{s_3}} = 0 \quad \text{in } \Omega, \\ u(x) > 0 \text{ in } \Omega, \quad u(x) = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a C^1 open bounded smooth domain with $0 \in \partial\Omega$ and $\partial\Omega$ is C^2 at 0 and the mean curvature $H(0) < 0$. The parameters satisfy

$$0 < s_2 < s_1 < 2, 0 \leq s_3 < 2, \lambda_2 > 0, 1 < p < 2^*(s_3) - 1.$$

Recall the following double critical problem

$$\begin{cases} \Delta u + \lambda \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 \quad \text{in } \Omega, \\ u(x) > 0 \text{ in } \Omega, \quad u(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

There has been a lot of papers concerning (1.2) under the premise of $s_2 < s_1$. We note that the case of $s_1 < s_2$ with $\lambda > 0$ is essentially the same. For the case of $s_1 = 2$ and $(i) N \geq 3, \lambda < (\frac{N-2}{2})^2, 0 < s_2 < s_1$ or $(ii) N \geq 4, 0 < \lambda < (\frac{N-2}{2})^2, s_2 = 0$, we refer to [5, 6, 7]. When $s_2 = 0$, equation (1.2) becomes

$$\Delta u + \lambda \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + u^{\frac{N+2}{N-2}} = 0 \text{ in } \Omega. \tag{1.3}$$

It is well known that (1.3) has no least-energy solution if $0 \leq s_1 < 2$ with $\lambda < 0$. However, for the case of $\lambda > 0, 0 < s_1 < 2$ and $s_2 = 0$, the existence of positive solution is proved by Hsia, Lin and Wadade [9]. In the very recent paper [11], the existence of positive solution for $N \geq 3, \lambda \in \mathbb{R}, 0 < s_2 < s_1 < 2$ is proved by Li and Lin. Basically, (1.2) has been studied for all the choices of the parameters s_1, s_2 under the premise that the coefficient of the highest power term is positive. However, an open problem is proposed by Li and Lin in [11, Remark 1.2] which says: *For the situation $s_1 < s_2$ and $\lambda < 0$, the existence of positive solutions to (1.2) is completely open. Even for the equation*

$$\Delta u - u^p + \frac{u^{2^*(s)-1}}{|x|^s} = 0 \textit{ in } \Omega, \tag{1.4}$$

where $0 < s < 2$ and $2^*(s) - 1 < p < \frac{N+2}{N-2}$, the existence problem still remains an interesting open question. It seems that the first partial answer to this open problem is obtained in [4].

Further, although (1.3) has no least-energy solutions for $\lambda < 0, 0 < s_1 < 2$, the following perturbed equation

$$\begin{cases} \Delta u - \frac{u^{2^*(s)-1}}{|x|^s} + u^{\frac{N+2}{N-2}} + u^p = 0 \text{ in } \Omega, \\ u(x) > 0 \text{ in } \Omega \text{ and } u(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{1.5}$$

has a positive solution if $N \geq 4, 2^*(s) - 1 < p < \frac{N+2}{N-2}$, see Li and Lin [11, Theorem 5.1].

In the current paper, we are interested in the more general perturbation problem than (1.5), that is, the equation (1.1). We obtain the following main theorems:

Theorem 1.1. *Suppose that Ω is an open bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$, $0 \in \partial\Omega$ and the mean curvature of $\partial\Omega$ at 0 is negative, i.e., $H(0) < 0$. Assume that $0 < s_2 < s_1 < 2, 0 \leq s_3 < 2, \lambda_2 > 0, 1 < p < 2^*(s_3) - 1$, and that one of the following conditions is satisfied:*

(1) $\lambda_1 > 0, \lambda_3 > 0.$

(2) $\lambda_1 > 0, \lambda_3 < 0, p \leq 2^*(s_1) - 1.$

(3) $\lambda_1 < 0, \lambda_3 > 0, p \geq 2^*(s_1) - 1.$

(4) $\lambda_1 < 0, \lambda_3 < 0, p < 2^*(s_2) - 1.$

Furthermore, if $\lambda_3 < 0$, we require either $p < \frac{N-s_3}{N-2}$ or $p \geq \frac{N-s_3}{N-2}$ with $|\lambda_3|$ small enough. Then (1.1) possesses a ground state solution.

Remark 1.1. *We remark that Theorem 1.1 does not cover the following two cases:*

- $\lambda_3 < 0, 2^*(s_1) - 1 < p \leq 2^*(s_3) - 1$ *and* $\lambda_1 > 0$*;*
- $\lambda_3 < 0, 2^*(s_2) - 1 < p \leq 2^*(s_3) - 1$ *and* $\lambda_1 < 0.$

Since for these cases, we do not know whether the (PS) sequence is bounded or not. In particular, the Nehari manifold method fails. The existence of the ground state solution for this two cases remains open.

However, when $|\lambda_3|$ small enough we may obtain the existence of positive solution. Precisely, we have the following result:

Theorem 1.2. *Suppose that Ω is an open bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$, $0 \in \partial\Omega$ and the mean curvature $H(0) < 0$. Assume that*

$$0 < s_2 < s_1 < 2, 0 \leq s_3 < 2, \lambda_2 > 0, \lambda_3 < 0$$

and that either $2^*(s_1) - 1 < p \leq 2^*(s_3) - 1$ if $\lambda_1 > 0$ or $2^*(s_2) - 1 < p \leq 2^*(s_3) - 1$ when $\lambda_1 < 0$. Then there exists $\lambda_0 < 0$ such that (1.1) has a positive solution for $\lambda_0 < \lambda_3 < 0$.

Remark 1.2. *In the above Theorem 1.2, we allow $p = 2^*(s_3) - 1$, it means that the equation (1.1) has three Hardy-Sobolev critical terms.*

This paper is organized as follows. In section 2, we give some properties of the Nehari manifold. Since the problem involves critical terms, it is well known that the lack of the compactness will bring much troubles. In section 3, we will determine the threshold of the functional for which the Palais-Smale condition holds and check that the ground state value lies in the safe region. Based on these preparations, we prove Theorem 1.1. In section 5, we will prove Theorem 1.2 by a perturbation method.

2 Nehari manifold

Let $L^p(\Omega, \frac{dx}{|x|^s})$ denote the space of L^p-integrable functions with respect to the measure $\frac{dx}{|x|^s}$. Let $|u|_{s,p} := \left(\int_\Omega \frac{|u|^p}{|x|^s}dx\right)^{\frac{1}{p}}$ and $|u|_p := |u|_{0,p}$. The Hardy-Sobolev inequality (see [2, 3, 8]) asserts that $D_0^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}(\mathbb{R}^N, \frac{dx}{|x|^s})$ is a continuous embedding for $s \in [0,2]$. That is, there exists $C_s > 0$ such that

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s}dx\right)^{\frac{2}{2^*(s)}} \leq C_s \int_{\mathbb{R}^N} |\nabla u|^2 dx \text{ for all } u \in D_0^{1,2}(\mathbb{R}^N). \tag{2.1}$$

If $|\Omega| < \infty$ and $p < 2^*(s_3) - 1$, we can obtain that $\displaystyle\int_\Omega \frac{|u|^{p+1}}{|x|^{s_3}}dx < \infty$ for all $u \in H_0^1(\Omega)$. In particular, the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega, \frac{dx}{|x|^{s_3}})$ is compact which was established in [13, Theorem 1.9] for $s_3 = 0$ and [4, Lemma 2.1] for $0 < s_3 < 2$. A function $u \in H_0^1(\Omega)$ is said to be a weak solution to the problem (1.1) iff

$$\int_\Omega \nabla u \cdot \nabla v dx - \lambda_1 \int_\Omega \frac{|u|^{2^*(s_1)-2}uv}{|x|^{s_1}}dx -$$
$$\lambda_2 \int_\Omega \frac{|u|^{2^*(s_2)-2}uv}{|x|^{s_2}}dx - \lambda_3 \int_\Omega \frac{|u|^{p-1}uv}{|x|^{s_3}}dx = 0 \tag{2.2}$$

for all $v \in H_0^1(\Omega)$. Thus, the corresponding energy functional of (1.1) is

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda_1}{2^*(s_1)}|u|_{s_1,2^*(s_1)}^{2^*(s_1)} - \frac{\lambda_2}{2^*(s_2)}|u|_{s_2,2^*(s_2)}^{2^*(s_2)} - \frac{\lambda_3}{p+1}|u|_{s_3,p+1}^{p+1}. \tag{2.3}$$

The associated Nehari manifold is defined as

$$\mathcal{N} := \left\{u \in H_0^1(\Omega)\backslash\{0\} : J(u) = 0\right\},$$

where

$$J(u) := \langle \Phi'(u), u\rangle = \|u\|^2 - \lambda_1|u|_{s_1,2^*(s_1)}^{2^*(s_1)} - \lambda_2|u|_{s_2,2^*(s_2)}^{2^*(s_2)} - \lambda_3|u|_{s_3,p+1}^{p+1} \tag{2.4}$$

and $\Phi'(u)$ denotes the Fréchet derivative of Φ at u; $\langle\cdot,\cdot\rangle$ is the dual product between $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$. We have the following properties on the Nehari manifold.

Lemma 2.1. *Assume that* $0 < s_2 < s_1 < 2, 0 \leq s_3 < 2, \lambda_2 > 0, 1 < p < 2^*(s_3) - 1$. *Then* $\forall\ u \in H_0^1(\Omega)\backslash\{0\}$, *there exists a unique* $t = t_u > 0$ *such that* $tu \in \mathcal{N}$ *if one of the following assumptions is satisfied:*

(1) $\lambda_1 > 0, \lambda_3 > 0$.

(2) $\lambda_1 > 0, \lambda_3 < 0, p \leq 2^*(s_1) - 1$.

(3) $\lambda_1 < 0, \lambda_3 > 0, p \geq 2^*(s_1) - 1$.

(4) $\lambda_1 < 0, \lambda_3 < 0, p < 2^*(s_2) - 1$.

Moreover, $\mathcal{N}$ is closed and bounded away from 0.

Proof. For any $u \in H_0^1(\Omega)$, we denote

$$a(u) := \|u\|^2,\ b(u) := |u|_{s_1, 2^*(s_1)}^{2^*(s_1)},\ c(u) := |u|_{s_2, 2^*(s_2)}^{2^*(s_2)},\ d(u) := |u|_{s_3, p+1}^{p+1}. \quad (2.5)$$

We will write them as a, b, c, d for simplicity if there is no ambiguity. Then, $\dfrac{d}{dt}\Phi(tu) = tg(t)$, where

$$g(t) := a - \lambda_1 b t^{2^*(s_1)-2} - \lambda_2 c t^{2^*(s_2)-2} - \lambda_3 d t^{p-1}.$$

We also see that for $t > 0, \frac{d}{dt}\Phi(tu) = 0$ if and only if $g(t) = 0$. Recalling that $\lambda_2 > 0, s_2 < s_1$ and $p < 2^*(s_2) - 1$ if $\lambda_3 < 0$, we obtain that $g(t) \to -\infty$ as $t \to +\infty$. Combine with $g(0) = a > 0$, we have that there exists some $t > 0$ such that $g(t) = 0$ due to the continuity of $g(t)$. It follows that $tu \in \mathcal{N}$. Let $u \in \mathcal{N}$, since $p > 1, 2^*(s_i) > 2$, by the embedding theorem we obtain that

$$a = \lambda_1 b + \lambda_2 c + \lambda_3 d \le C\Big(a^{\frac{2^*(s_1)}{2}} + a^{\frac{2^*(s_2)}{2}} + a^{\frac{p+1}{2}}\Big),$$

which implies that there exists some $\delta_0 > 0$ such that

$$\|u\| = a^{\frac{1}{2}} \ge \delta_0 \text{ for all } u \in \mathcal{N}. \quad (2.6)$$

Then for any $u \neq 0$, $t_0 := \inf\{t | g(t) = 0\} > 0$ and by the continuity of $g(t)$, we obtain that $g(t_0) = 0$. Without loss of generality, we may assume that $t_0 = 1$, that is, $g(1) = 0$ and $g(t) > 0$ for all $t \in (0, 1)$.

If $\lambda_1 > 0, \lambda_3 > 0$, it is easy to see that $g'(t) < 0$. Hence, $g(t) < g(1) = 0$ for all $t > 1$.

If $\lambda_1 > 0, \lambda_3 < 0, p \le 2^*(s_1) - 1$, we consider $t > 1$ first. We have $g'(t) = -t^{p-2}h(t)$, where

$$h(t) := \lambda_1 b\big(2^*(s_1) - 2\big)t^{2^*(s_1)-p-1} + \lambda_2 c\big(2^*(s_2) - 2\big)t^{2^*(s_2)-p-1} + \lambda_3 d(p-1).$$

Recall that $a - \lambda_1 b - \lambda_2 c - \lambda_3 d = 0$, we obtain that

$$\begin{aligned} h(t) >& \lambda_1 b\big(2^*(s_1) - 2\big) + \lambda_2 c\big(2^*(s_2) - 2\big) + \lambda_3 d(p-1) \\ =& \lambda_1 b\big(2^*(s_1) - p - 1\big) + \lambda_2 c\big(2^*(s_2) - p - 1\big) + a(p-1) > 0. \end{aligned} \quad (2.7)$$

It follows that $g'(t) < 0$ for all $t > 1$ and then $g(t) < g(1) = 0$ for all $t > 1$.

If $\lambda_1 < 0, \lambda_3 > 0, 2^*(s_1) - 1 \le p$, then we have $g'(t) = -t^{2^*(s_1)-3}q(t)$, where

$$q(t) := \lambda_1\big(2^*(s_1) - 2\big)b + \lambda_2\big(2^*(s_2) - 2\big)ct^{2^*(s_2)-2^*(s_1)} + \lambda_3(p-1)dt^{p+1-2^*(s_1)}.$$

Assume $t > 1$, we obtain that

$$\begin{aligned} q(t) >& \lambda_1\big(2^*(s_1)-2\big)b + \lambda_2\big(2^*(s_2)-2\big)c + \lambda_3(p-1)d \\ =& a\big(2^*(s_1)-2\big) + \lambda_2 c\big(2^*(s_2)-2^*(s_1)\big) + \lambda_3 d\big(p+1-2^*(s_1)\big) \\ >& 0. \end{aligned} \tag{2.8}$$

Hence, we also obtain that $g'(t) < 0$ for $t > 1$.

If $\lambda_1 < 0, \lambda_3 < 0, p < 2^*(s_2) - 1$, we have

$$tg'(t) = -\lambda_1 b\big(2^*(s_1)-2\big)t^{2^*(s_1)-2} - \lambda_3 d(p-1)t^{p-1} - \lambda_2 c\big(2^*(s_2)-2\big)t^{2^*(s_2)-2}.$$

Recalling that $\lambda_2 c = a - \lambda_1 b - \lambda_3 d$, we obtain that

$$\begin{aligned} tg'(t) =& \Big[\big(2^*(s_2)-2\big)t^{2^*(s_2)-2^*(s_1)} - \big(2^*(s_1)-2\big)\Big]\lambda_1 b t^{2^*(s_1)-2} \\ &+ \Big[\big(2^*(s_2)-2\big)t^{2^*(s_2)-p-1} - (p-1)\Big]\lambda_3 d t^{p-1} \\ &- a\big(2^*(s_2)-2\big)t^{2^*(s_2)-2} \\ =:& I + II + III. \end{aligned}$$

Since $s_2 < s_1$, we have $2^*(s_2) > 2^*(s_1)$. Thus, for $t > 1$, we obtain

$$\big(2^*(s_2)-2\big)t^{2^*(s_2)-2^*(s_1)} - \big(2^*(s_1)-2\big) > 2^*(s_2) - 2^*(s_1) > 0.$$

It follows that $I < 0$ due to the fact of $\lambda_1 < 0$. Similarly, since $p < 2^*(s_2) - 1$, we can prove that $II < 0$ for $t > 1$. Obviously, $III < 0$. We deduce that $g'(t) < 0$ for all $t > 1$. Based on the above arguments, we obtain that $g(t) < g(1) = 0$ for all $t > 1$. Hence, for any $0 \neq u \in H_0^1(\Omega)$, there exists a unique $t > 0$ denoted by t_u such that $t_u u \in \mathcal{N}$. By (2.6), we have that $\mathcal{N}$ is bounded away from 0 and that $\mathcal{N}$ is closed. □

Lemma 2.2. *Under the assumptions of Lemma 2.1, any $(PS)_c$ sequence $\{u_n\}$ of $\Phi(u)$, i.e., $\Phi(u_n) \to c,\ \ \Phi'(u_n) \to 0$ in $H^{-1}(\Omega)$, is bounded in $H_0^1(\Omega)$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a $(PS)_c$ sequence of $\Phi(u)$, then we have

$$\Phi(u_n) = \frac{1}{2}a(u_n) - \frac{\lambda_1}{2^*(s_1)}b(u_n) - \frac{\lambda_2}{2^*(s_2)}c(u_n) - \frac{\lambda_3}{p+1}d(u_n) = c + o(1) \tag{2.9}$$

and $\langle \Phi'(u_n), u_n\rangle = a(u_n) - \lambda_1 b(u_n) - \lambda_2 c(u_n) - \lambda_3 d(u_n) = o(1)\|u_n\|$, where $a(u), b(u), c(u), d(u)$ are defined by (2.5).

Case 1. Assume $\lambda_1 > 0, \lambda_3 > 0$. If $p + 1 \geq 2^*(s_1)$, we have that

$$c + o(1)(1 + \|u_n\|) = \Phi(u_n) - \frac{1}{2^*(s_1)}\langle \Phi'(u_n), u_n\rangle \geq (\frac{1}{2} - \frac{1}{2^*(s_1)})\|u_n\|^2.$$

If $p + 1 < 2^*(s_1)$, note that $s_2 < s_1, p + 1 < 2^*(s_1)$, it follows that

$$c + o(1)(1 + \|u_n\|) = \Phi(u_n) - \frac{1}{p+1}\langle \Phi'(u_n), u_n\rangle \geq (\frac{1}{2} - \frac{1}{p+1})\|u_n\|^2,$$

Case 2. If $\lambda_1 > 0, \lambda_3 < 0, p \le 2^*(s_1) - 1$, we have

$$c + o(1) = \Phi(u_n) = (\frac{1}{2} - \frac{1}{p+1})a(u_n) + (\frac{1}{p+1} - \frac{1}{2^*(s_1)})\lambda_1 b(u_n) + (\frac{1}{p+1} - \frac{1}{2^*(s_2)})\lambda_2 c(u_n) + o(1)\|u_n\|.$$

Hence, $c + o(1)(1 + \|u_n\|) \ge (\frac{1}{2} - \frac{1}{p+1})\|u_n\|^2$.
Case 3. If $\lambda_1 < 0, \lambda_3 > 0, 2^*(s_1) - 1 \le p$, then

$$\begin{aligned} c + o(1) =& \Phi(u_n) = \big(\frac{1}{2} - \frac{1}{2^*(s_1)}\big)a(u_n) + \big(\frac{1}{2^*(s_1)} - \frac{1}{2^*(s_2)}\big)\lambda_2 c(u_n) \\ &+ \big(\frac{1}{2^*(s_1)} - \frac{1}{p+1}\big)\lambda_3 d(u_n) + o(1)\|u_n\|. \end{aligned}$$

Hence, $c + o(1)(1 + \|u_n\|) \ge \big(\frac{1}{2} - \frac{1}{2^*(s_1)}\big)\|u_n\|^2$.
Case 4. If $\lambda_1 < 0, \lambda_3 < 0, p < 2^*(s_2) - 1$, similarly we have

$$c + o(1)(1 + \|u_n\|) \ge \big(\frac{1}{2} - \frac{1}{2^*(s_2)}\big)\|u_n\|^2.$$

Based on the above arguments, we can see that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. □

Remark 2.1. *Under the assumptions of Lemma 2.1, we define*

$$c_0 := \inf_{u \in \mathcal{N}} \Phi(u) \tag{2.10}$$

and

$$\eta := \min\{\frac{1}{2} - \frac{1}{p+1}, \frac{1}{2} - \frac{1}{2^*(s_1)}\} > 0.$$

Similar to the prove of Lemma 2.2, we see that $c_0 \ge \eta\delta_0^2 > 0$, where δ_0 is given by (2.6). If c_0 is achieved by some $u \in \mathcal{N}$, then u is a ground state solution of (1.1).

Lemma 2.3. *Under the assumptions of Lemma 2.1, let $\{u_n\} \subset \mathcal{N}$ be a $(PS)_c$ sequence for $\Phi\big|_{\mathcal{N}}$, that is, $\Phi(u_n) \to c$ and $\Phi'\big|_{\mathcal{N}}(u_n) \to 0$ in $H^{-1}(\Omega)$. Then $\{u_n\}$ is also a $(PS)_c$ sequence for Φ.*

Proof. Firstly, by the similar arguments as that in Lemma 2.2, we may show that $\{(u_n, v_n)\}$ is bounded in $\mathscr{D}$. Let $\{t_n\} \subset \mathbb{R}$ be a sequence of multipliers satisfying

$$\Phi'(u_n) = \Phi'\big|_{\mathcal{N}}(u_n) + t_n J'(u_n).$$

Testing by u_n, we obtain that $t_n\langle J'(u_n), u_n\rangle \to 0$. Recalling that for any $u \in \mathcal{N}$, we have

$$\langle J'(u), u\rangle = 2a - 2^*(s_1)\lambda_1 b - 2^*(s_2)\lambda_2 c - (p+1)\lambda_3 d$$

and $a - \lambda_1 b - \lambda_2 c - \lambda_3 d = 0$, where a, b, c, d is defined by (2.5).

(1) If $\lambda_1 > 0, \lambda_3 > 0$, then

$$\begin{aligned}\langle J'(u), u\rangle =& 2a - 2^*(s_1)\lambda_1 b - 2^*(s_2)\lambda_2 c - (p+1)\lambda_3 d \\ <& -\min\{2^*(s_1)-2, 2^*(s_2)-2, p-1\}(\lambda_1 b + \lambda_2 c + \lambda_3 d) \\ =& -\min\{2^*(s_1)-2, 2^*(s_2)-2, p-1\}a.\end{aligned}$$

(2) If $\lambda_1 > 0, \lambda_3 < 0, p \le 2^*(s_1) - 1$, we have $p < 2^*(s_2) - 1$ and then

$$\langle J'(u), u\rangle = 2a - 2^*(s_1)\lambda_1 b - 2^*(s_2)\lambda_2 c - (p+1)(a - \lambda_1 b - \lambda_2 c) < -(p-1)a.$$

(3) If $\lambda_1 < 0, \lambda_3 > 0, 2^*(s_1) - 1 \le p$, we also have

$$\begin{aligned}\langle J'(u), u\rangle =& \big(2 - 2^*(s_1)\big)a + \big(2^*(s_1) - 2^*(s_2)\big)\lambda_2 c + \big(2^*(s_1) - p - 1\big)\lambda_3 d \\ <& -\big(2^*(s_1) - 2\big)a.\end{aligned}$$

(4) If $\lambda_1 < 0, \lambda_3 < 0, p < 2^*(s_2) - 1$, we have

$$\begin{aligned}\langle J'(u), u\rangle =& \big(2 - 2^*(s_2)\big)a + \big(2^*(s_2) - 2^*(s_1)\big)\lambda_1 b + \big(2^*(s_2) - p - 1\big)\lambda_3 d \\ <& -\big(2^*(s_2) - 2\big)a.\end{aligned}$$

Thus, under the assumptions of Lemma 2.1, $\langle J'(u), u\rangle < -\varrho\|u\|^2$ for all $u \in \mathcal{N}$, where $\varrho := \min\{2^*(s_1)-2, 2^*(s_2)-2, p-1\} > 0$. Invoke (2.6), we have

$$\langle J'(u_n), u_n\rangle < -\varrho\delta_0^2 \text{ for all } n.$$

Hence, we obtain that t_n is bounded. On the other hand, it is easy to see that $\langle J'(u_n), u_n\rangle$ is bounded due to the boundedness of $\{u_n\}$. We claim that $t_n \to 0$. If not, up to a subsequence, we may assume that $t_n \to t_0 \ne 0$ and $\langle J'(u_n), u_n\rangle \to d_0 < -\varrho\delta_0^2$. Then

$$\big|t_n\langle J'(u_n), u_n\rangle\big| \to |t_0 d_0| > |t_0|\varrho\delta_0^2 \ne 0,$$

a contradiction. Thus, we see that $t_n \to 0$ and it follows that $\Phi'(u_n) \to 0$ in $H^{-1}(\Omega)$. □

3 Analysis of the Palais-Smale sequences

Understanding asymptotic behavior is usually fundamental in the resolution of mathematical problems, particularly the problem possesses critical terms. The following result is due to [11]:

Theorem A. ([11, Theorem 1.2]) *Let $N \ge 3, 0 < s_2 < s_1 < 2, \lambda \in \mathbb{R}$, then the following problem*

$$\begin{cases}\Delta u + \lambda\frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 & \textit{in } \mathbb{R}^N_+, \\ u(x) > 0 \quad \textit{in } \Omega, \quad u(x) = 0 & \textit{on } \partial\mathbb{R}^N_+,\end{cases} \tag{3.1}$$

has a least-energy solution $u \in H_0^1(\mathbb{R}_+^N)$. $\square$

Let $u > 0$ be the least energy solution of (3.1), then

$$|u(y)| \leq C|y|^{1-N} \quad \text{for } |y| \geq 1 \text{ and } |\nabla u(y)| \leq |y|^{-N} \quad \text{for } |y| \geq 1. \tag{3.2}$$

See [11, Page 16-17]. We also note that, by the well-known moving plane method, one can prove that $u(x', x_N)$ is axially symmetric with respect to the x_N-axis, i.e., $u(x', x_N) = u(|x'|, x_N)$, where $x' = (x_1, \cdots, x_{N-1})$. Since the argument is standard, we omit the proof here (see [12, Lemma 2.6]). When $\lambda_2 > 0$, define $v = \lambda_2^{\frac{1}{2^*(s_2)-2}} u$, a direct calculation shows that u is a solution of

$$\begin{cases} \Delta u + \lambda_1 \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \lambda_2 \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 & \text{in } \mathbb{R}_+^N, \\ u(x) > 0 \quad \text{in } \mathbb{R}_+^N, \quad u(x) = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \tag{3.3}$$

if and only if v is a solution to (3.1) with $\lambda = \lambda_1 \lambda_2^{\frac{2-2^*(s_1)}{2^*(s_2)-2}}$. We denote the least energy corresponding to (3.3) by c_{λ_1,λ_2}, that is,

$$c_{\lambda_1,\lambda_2} = \inf\{A_{\lambda_1,\lambda_2}(u) | u \text{ is a solution to } (3.3)\},$$

where

$$A_{\lambda_1,\lambda_2}(u) := \frac{1}{2}\int_{\mathbb{R}_+^N} |\nabla u|^2 dx - \frac{\lambda_1}{2^*(s_1)} \int_{\mathbb{R}_+^N} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx - \frac{\lambda_2}{2^*(s_2)} \int_{\mathbb{R}_+^N} \frac{|u|^{2^*(s_2)}}{|x|^{s_2}} dx.$$

It is easy to see that

$$A_{\lambda,1}(v) = \lambda_2^{\frac{2}{2^*(s_2)-2}} A_{\lambda_1,\lambda_2}(u), \tag{3.4}$$

where

$$v = \lambda_2^{\frac{1}{2^*(s_2)-2}} u, \lambda = \lambda_1 \lambda_2^{\frac{2-2^*(s_1)}{2^*(s_2)-2}}.$$

It follows that

$$c_{\lambda_1,\lambda_2} = \lambda_2^{\frac{-2}{2^*(s_2)-2}} c_{\lambda,1}, \quad \lambda = \lambda_1 \lambda_2^{\frac{2-2^*(s_1)}{2^*(s_2)-2}}. \tag{3.5}$$

Let $w > 0$ be a ground state solution to (3.3), then

$$|w(y)| \leq C|y|^{1-N} \quad \text{for } |y| \geq 1 \tag{3.6}$$

and

$$|\nabla w(y)| \leq \lambda_2^{-\frac{1}{2^*(s_2)-2}} |y|^{-N} \quad \text{for } |y| \geq 1. \tag{3.7}$$

Similar to [4, Theorem 3.1], we can establish the following splitting result which provides a precise description of a behavior of $(PS)_c$ sequence for $\Phi(u)$.

Theorem 3.1. *(Splitting Theorem) Suppose that* $\{u_n\} \subset H_0^1(\Omega)$ *is a bounded* $(PS)_c$ *sequence of the functional* $\Phi(u)$*. That is,* $\Phi(u_n) \to c$ *and* $\Phi'(u_n) \to 0$ *strongly in* $H^{-1}(\Omega)$ *as* $n \to \infty$*. Then there exists a solution* U^0 *to the equation in (1.1) (*$U^0 \equiv 0$ *is allowed), number* $k \in \mathbb{N} \cup \{0\}$*,* k *functions* $U^1, \cdots, U^k$ *and* k *sequences of radius* $r_n^j > 0, 1 \leq j \leq k$ *such that the following properties are satisfied up to a subsequence if necessary: Either*

(a) $u_n \to U^0$ *in* $H_0^1(\Omega)$ *or*

(b) *the following items all are true:*

(b1) $U^j \in D^{1,2}(\mathbb{R}_+^N) \subset D^{1,2}(\mathbb{R}^N)$ *are nontrivial solutions of (3.3);*

(b2) $r_n^j \to 0$ *as* $n \to \infty$*;*

(b3) $\|u_n - U^0 - \sum_{j=1}^k (r_n^j)^{\frac{2-N}{2}} U^j(\frac{\cdot}{r_n^j})\| \to 0$, *where* $\|\cdot\|$ *is the norm in* $D^{1,2}(\mathbb{R}^N)$*;*

(b4) $\|u_n\|^2 \to \|U^0\|^2 + \sum_{j=1}^k \|U^j\|^2$*;*

(b5) $\Phi(u_n) \to \Phi(U^0) + \sum_{j=1}^k A_{\lambda_1,\lambda_2}(U^j)$, *where*

$$A_{\lambda_1,\lambda_2}(u) := \frac{1}{2}\int_{\mathbb{R}_+^N} |\nabla u|^2 dx - \frac{\lambda_1}{2^*(s_1)}\int_{\mathbb{R}_+^N} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx$$
$$-\frac{\lambda_2}{2^*(s_2)}\int_{\mathbb{R}_+^N} \frac{|u|^{2^*(s_2)}}{|x|^{s_2}} dx.$$

□

The following corollary is a straightforward consequence of the above theorem.

Corollary 3.1. *Under the assumptions of Lemma 2.1, the functional* $\Phi(u)$ *satisfies* $(PS)_c$ *condition for* $c < c_{\lambda_1,\lambda_2}$.

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be such that $\Phi(u_n) \to c < c_{\lambda_1,\lambda_2}$, $\Phi'(u_n) \to 0$ in $H^{-1}(\Omega)$. By Lemma 2.2, we obtain that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By Theorem 3.1, we obtain that $u_n \to U^0$ in $H_0^1(\Omega)$ up to a subsequence. If not, $k \neq 0$ and

$$\sum_{j=1}^k A_{\lambda_1,\lambda_2}(U^j) \geq c_{\lambda_1,\lambda_2}.$$

Recalling that $c_0 > 0$ (see Remark 2.1), we have $\Phi(U^0) \geq 0$, then

$$c = \Phi(u_n) + o(1) = \Phi(U^0) + \sum_{j=1}^k A_{\lambda_1,\lambda_2}(U^j) \geq c_{\lambda_1,\lambda_2},$$

a contradiction. □

Before giving the proof of Theorem 3.1, we need to prepare the following two auxiliary results:

Lemma 3.1. *Let* $\{u_n\} \subset H_0^1(\Omega)$ *be such that* $A_{\lambda_1,\lambda_2}(u_n) \to c$ *and* $A'_{\lambda_1,\lambda_2}(u_n) \to 0$ *in* $H^{-1}(\Omega)$. *For* $\{r_n\} \subset (0,\infty)$ *with* $r_n \to 0$, *let* $v_n(x) := r_n^{\frac{N-2}{2}} u_n(r_n x)$ *be such that* $v_n \rightharpoonup v$ *in* $D^{1,2}(\mathbb{R}^N)$ *and* $v_n \to v$ *a.e. on* $\mathbb{R}^N$. *Then,* $A'_{\lambda_1,\lambda_2}(v) = 0$ *and the sequence*

$$w_n(x) := u_n(x) - r_n^{\frac{2-N}{2}} v(\frac{x}{r_n})$$

satisfies $A_{\lambda_1,\lambda_2}(w_n) \to c - A_{\lambda_1,\lambda_2}(v)$, $A'_{\lambda_1,\lambda_2}(w_n) \to 0$ *in* $H^{-1}(\Omega)$ *and* $\|w_n\|^2 = \|u_n\|^2 - \|v\|^2 + o(1)$.

Proof. Without loss of generality, we assume that $\partial R^N_+ := \{x_N = 0\}$ is tangent to $\partial\Omega$ at 0, and that $-e_N = (0, \cdots, -1)$ is the outward normal to $\partial\Omega$ at that point. For any compact $K \subset \mathbb{R}^N_-$, we have for n large enough, that $\frac{\Omega}{r_n} \cap K = \emptyset$ as $r_n \to 0$. Since $supp\ v_n \subset \frac{\Omega}{r_n}$ and $v_n \to v$ a.e. in $\mathbb{R}^N$, it follows that $v = 0$ a.e. on K. Therefore, $supp\ v \subset \mathbb{R}^N_+$. Hence, for n large enough, we obtain that $supp\ v_n \subset \mathbb{R}^N_+$ and $v_n \rightharpoonup v$ in $D^{1,2}_0(\mathbb{R}^N_+)$. We note that the functional A_{λ_1,λ_2} is invariant under dilation, hence,

$$\|v_n\|^2 = \int_{\mathbb{R}^N} |\nabla(r_n^{\frac{N-2}{2}} u_n(r_n x))|^2 dx = \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \|u_n\|^2$$

and

$$\int_{\mathbb{R}^N} \frac{|v_n|^{2^*(s_i)}}{|x|^{s_i}} dx = \int_{\mathbb{R}^N} r_n^{N-s_i} \frac{|u_n(r_n x)|^{2^*(s_i)}}{|x|^{s_i}} dx = \int_{\mathbb{R}^N} \frac{|u_n|^{2^*(s_i)}}{|x|^{s_i}} dx.$$

Similarly, we have that $\|w_n\|^2 = \|(r_n)^{\frac{N-2}{2}} w_n(r_n x)\|^2$. Notice that $(r_n)^{\frac{N-2}{2}} w_n(r_n x) = v_n - v$. When $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$, we have

$$\|w_n\|^2 = \|v_n - v\|^2 = \|v_n\|^2 - \|v\|^2 + o(1) = \|u_n\|^2 - \|v\|^2 + o(1).$$

Recalling that $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$, by Brezis-Lieb type lemma (see [1] for s=0 and [6] for $s > 0$) and the invariance property again, we have

$$\begin{aligned}
&A_{\lambda_1,\lambda_2}(w_n) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w_n|^2 dx - \frac{\lambda_1}{2^*(s_1)} \int_{\mathbb{R}^N} \frac{|w_n|^{2^*(s_1)}}{|x|^{s_1}} dx - \frac{\lambda_2}{2^*(s_2)} \int_{\mathbb{R}^N} \frac{|w_n|^{2^*(s_2)}}{|x|^{s_2}} dx \\
&= A_{\lambda_1,\lambda_2}\big(r_n^{\frac{N-2}{2}} w_n(r_n x)\big) \\
&= A_{\lambda_1,\lambda_2}(v_n - v) \\
&= A_{\lambda_1,\lambda_2}(v_n) - A_{\lambda_1,\lambda_2}(v) + o(1) \\
&= A_{\lambda_1,\lambda_2}(u_n) - A_{\lambda_1,\lambda_2}(v) + o(1) \\
&= c - A_{\lambda_1,\lambda_2}(v) + o(1).
\end{aligned}$$

For any $h \in C^\infty_0(\mathbb{R}^N_+)$, let $h_n(x) := (r_n)^{\frac{2-N}{2}} h(\frac{x}{r_n})$, then we have that $h_n \in H^1_0(\Omega)$ for n large enough due to the assumption that $r_n \to 0$. Thus

$$\begin{aligned}
\langle A'_{\lambda_1,\lambda_2}(v), h\rangle &= \langle A'_{\lambda_1,\lambda_2}(v_n), h\rangle + o(1) \\
&= \langle A'_{\lambda_1,\lambda_2}(u_n), h_n\rangle + o(1) \\
&= o(1)\|h_n\| + o(1) \\
&= o(1)\|h\| + o(1),
\end{aligned}$$

which implies that $A'_{\lambda_1,\lambda_2}(v) = 0$. For any $h \in H^1_0(\Omega)$, let $\tilde{h}_n(x) := r_n^{\frac{N-2}{2}} h(r_n x)$. Then for n large enough, $supp\ \tilde{h}_n \subset \mathbb{R}^N_+$. By the Brezis-Lieb type lemma again, we obtain that

$$A'_{\lambda_1,\lambda_2}(v_n) - A'_{\lambda_1,\lambda_2}(v_n - v) - A'_{\lambda_1,\lambda_2}(v) \to 0 \text{ in } H^{-1}(\mathbb{R}^N). \tag{3.8}$$

Hence, for any $h \in H_0^1(\Omega)$,

$$
\begin{aligned}
&\langle A'_{\lambda_1,\lambda_2}(w_n), h\rangle = \langle A'_{\lambda_1,\lambda_2}(r_n^{\frac{N-2}{2}} w_n(r_n x)), \tilde{h}_n(x)\rangle \\
&= \langle A'_{\lambda_1,\lambda_2}(r_n^{\frac{N-2}{2}} w_n(r_n x)), \tilde{h}_n(x)\rangle + \langle A'_{\lambda_1,\lambda_2}(v(x)), \tilde{h}_n(x)\rangle \text{ (since } A'_{\lambda_1,\lambda_2}(v) = 0) \\
&= \langle A'_{\lambda_1,\lambda_2}(v_n - v), \tilde{h}_n(x)\rangle + \langle A'_{\lambda_1,\lambda_2}(v(x)), \tilde{h}_n(x)\rangle \\
&= \langle A'_{\lambda_1,\lambda_2}(v_n), \tilde{h}_n(x)\rangle + o(1)\|\tilde{h}_n\| \quad \text{(by (3.8))} \\
&= \langle A'_{\lambda_1,\lambda_2}(u_n), h(x)\rangle + o(1)\|\tilde{h}_n\| \\
&= o(1)\|h\| \quad (\text{since } \|\tilde{h}_n\| \equiv \|h\|).
\end{aligned}
$$

□

Lemma 3.2. *(See [6, Lemma 3.5]) If $u \in D^{1,2}(\mathbb{R}^N)$ and $h \in C_0^\infty(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \frac{h^2|u|^{2^*(s)}}{|x|^s} dx \le \mu_s(\mathbb{R}^N)^{-1}\Big(\int_{supp\ h} \frac{|u|^{2^*(s)}}{|x|^s}\Big)^{\frac{2^*(s)-2}{2^*(s)}} \int_{\mathbb{R}^N} |\nabla(hu)|^2 dx,$$

where

$$\mu_s(\mathbb{R}^N) := \inf\Big\{\int_{\mathbb{R}^N} |\nabla u|^2 dx : \quad u \in D_0^{1,2}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s} < \infty\Big\}. \tag{3.9}$$

Proof of Theorem 3.1. Let $\{u_n\} \subset H_0^1(\Omega)$ be a bounded $(PS)_c$ sequence of $\Phi(u)$. Up to a subsequence, there is an $U^0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup U^0$ in $H_0^1(\Omega)$ and $\nabla u_n \to \nabla U^0$ a.e. on $\mathbb{R}^N$. Evidently, $\Phi'(U^0) = 0$. Moreover, the sequence $u_n^1 := u_n - U^0$ satisfies

$$\begin{cases} \|u_n^1\| = \|u_n\|^2 - \|U^0\|^2 + o(1), \\ A'_{\lambda_1,\lambda_2}(u_n^1) \to 0 \quad \text{in } H^{-1}(\Omega), \\ A_{\lambda_1,\lambda_2}(u_n^1) \to c - \Phi(U^0). \end{cases} \tag{3.10}$$

If $u_n^1 \to 0$ in $H_0^1(\Omega)$, we are done. If not, it is easy to see that

$$\eta_0 := \liminf_{n\to\infty}\Big(\lambda_1 \int_\Omega \frac{|u_n^1|^{2^*(s_1)}}{|x|^{s_1}} dx + \lambda_2 \int_\Omega \frac{|u_n^1|^{2^*(s_2)}}{|x|^{s_2}} dx\Big) > 0. \tag{3.11}$$

For the case of $\lambda_1 > 0$, we define an analogue of Levy's concentration function

$$Q_n(r) := \int_{B(0,r)} \Big(\lambda_1 \frac{|u_n^1|^{2^*(s_1)}}{|x|^{s_1}} + \lambda_2 \frac{|u_n^1|^{2^*(s_2)}}{|x|^{s_2}}\Big) dx.$$

Since $Q_n(0) = 0$ and $Q_n(\infty) \ge \eta_0 > 0$, there exists a sequence $r_n^1 > 0$ such that for each n

$$\delta = \int_{B(0,r_n^1)} \Big(\lambda_1 \frac{|u_n^1|^{2^*(s_1)}}{|x|^{s_1}} + \lambda_2 \frac{|u_n^1|^{2^*(s_2)}}{|x|^{s_2}}\Big) dx, \tag{3.12}$$

here we take δ so small that

$$\lambda_1^{\frac{2}{2^*(s_1)}} \mu_{s_1}(\mathbb{R}^N)^{-1} \delta^{\frac{2^*(s_1)-2}{2^*(s_1)}} + \lambda_2^{\frac{2}{2^*(s_2)}} \mu_{s_2}(\mathbb{R}^N)^{-1} \delta^{\frac{2^*(s_2)-2}{2^*(s_2)}} < \frac{1}{2}, \tag{3.13}$$

where $\mu_s(\mathbb{R}^N)$ is defined by (3.9). Define $v_n^1(x) := (r_n^1)^{\frac{N-2}{2}} u_n^1(r_n^1 x)$. Since $\|v_n^1\| = \|u_n^1\|$ is bounded, we may assume $v_n^1 \rightharpoonup U^1$ in $D^{1,2}(\mathbb{R}^N)$, $v_n^1 \to U^1$ a.e. on $\mathbb{R}^N$ and

$$\delta = \int_{B(0,1)} \Big(\lambda_1 \frac{|v_n^1|^{2^*(s_1)}}{|x|^{s_1}} + \lambda_2 \frac{|v_n^1|^{2^*(s_2)}}{|x|^{s_2}}\Big) dx.$$

Next, we show that $U^1 \not\equiv 0$. Define $\Omega_n = \frac{1}{r_n^1}\Omega$ and let $f_n \in H_0^1(\Omega)$ be such that for any $h \in H_0^1(\Omega)$, we have $\langle A'_{\lambda_1,\lambda_2}(u_n^1), h\rangle = \int_\Omega \nabla f_n \cdot \nabla h$. Then $g_n(x) := (r_n^1)^{\frac{N-2}{2}} f_n(r_n^1 x)$ satisfies $\int_{\Omega_n} |\nabla g_n|^2 = \int_\Omega |\nabla f_n|^2$ and $\langle A'_{\lambda_1,\lambda_2}(v_n^1), h\rangle = \int_{\Omega_n} \nabla g_n \cdot \nabla h$ for any $h \in H_0^1(\Omega_m)$. If $U_1 \equiv 0$, then for any $h \in C_0^\infty(\mathbb{R}^N)$ with $supp\ h \subset B(0,1)$, from Lemma 3.2 and the fact of (3.13), we get that

$$\begin{aligned}
&\int_{B(0,1)} |\nabla(h v_n^1)|^2 \\
&= \int_{B(0,1)} \nabla v_n^1 \cdot \nabla(h^2 v_n^1) + o(1) \\
&= \lambda_1 \int \frac{h^2 |v_n^1|^{2^*(s_1)}}{|x|^{s_1}} + \lambda_2 \int \frac{h^2 |v_n^1|^{2^*(s_2)}}{|x|^{s_2}} + \int \nabla g_n \cdot \nabla(h^2 v_n^1) + o(1) \\
&= \lambda_1 \int \frac{h^2 |v_n^1|^{2^*(s_1)}}{|x|^{s_1}} + \lambda_2 \int \frac{h^2 |v_n^1|^{2^*(s_2)}}{|x|^{s_2}} + \langle A'_{\lambda_1,\lambda_2}(v_n), h^2 v_n^1\rangle + o(1) \\
&\le \lambda_1 \mu_{s_1}(\mathbb{R}^N)^{-1} \Big(\int_{B(0,1)} \frac{|u_n^1|^{2^*(s_1)}}{|x|^{s_1}}\Big)^{\frac{2-2^*(s_1)}{2}} \int |\nabla(h v_n^1)|^2 \\
&\quad + \lambda_2 \mu_{s_2}(\mathbb{R}^N)^{-1} \Big(\int_{B(0,1)} \frac{|u_n^1|^{2^*(s_2)}}{|x|^{s_2}}\Big)^{\frac{2-2^*(s_2)}{2}} \int |\nabla(h v_n^1)|^2 + o(1) \\
&\le \frac{1}{2} \int |\nabla(h v_n^1)|^2 + o(1).
\end{aligned}$$

Hence, $\nabla v_n^1 \to 0$ in $L^2_{loc}\big(B(0,1)\big)$ and $v_n^1 \to 0$ in $L^{2^*(s_i)}_{loc}\big(B(0,1), |x|^{-s_i} dx\big)$, which contradicts the fact that

$$\int_{B(0,1)} \Big(\lambda_1 \frac{|v_n^1|^{2^*(s_1)}}{|x|^{s_1}} + \lambda_2 \frac{|v_n^1|^{2^*(s_2)}}{|x|^{s_2}}\Big) dx = \delta > 0.$$

Thus we have proved that $U^1 \not\equiv 0$. Apply the similar argument for the case of $\lambda_1 < 0$ with a modified concentration function

$$Q_n(r) := \int_{B(0,r)} \lambda_2 \frac{|u_n^1|^{2^*(s_2)}}{|x|^{s_2}} dx.$$

In this case ,we take $0 < \delta < \big(\frac{\mu_{s_2}(\mathbb{R}^N)}{2}\big)^{\frac{N-s_2}{2-s_2}}$ small enough and a sequence $r_n^1 > 0$ with $Q_{r_n^1}(x) = \delta$. We also define $v_n^1(x) := (r_n^1)^{\frac{N-2}{2}} u_n^1(r_n^1 x)$ and assume that $v_n^1 \rightharpoonup$

U^1 in $D^{1,2}(\mathbb{R}^N)$, $v_n^1 \to U^1$ a.e. on $\mathbb{R}^N$ and

$$\delta = \int_{B(0,1)} \lambda_2 \frac{|v_n^1|^{2^*(s_2)}}{|x|^{s_2}} dx.$$

Next we will prove that $U^1 \not\equiv 0$ for this case.

If $U_1 \equiv 0$, choose any $h \in C_0^\infty(\mathbb{R}^N)$ such that $supp\ h \subset B(0,1)$ and invoke Lemma 3.2 and the fact of $0 < \delta < \big(\frac{\mu_{s_2}(\mathbb{R}^N)}{2}\big)^{\frac{N-s_2}{2-s_2}}$, we have the following estimates:

$$\begin{aligned}
&\int_{B(0,1)} |\nabla(hv_n^1)|^2 \\
&= \int_{B(0,1)} \nabla v_n^1 \cdot \nabla(h^2 v_n^1) + o(1) \\
&= \lambda_1 \int \frac{h^2|v_n^1|^{2^*(s_1)}}{|x|^{s_1}} + \lambda_2 \int \frac{h^2|v_n^1|^{2^*(s_2)}}{|x|^{s_2}} + \int \nabla g_n \cdot \nabla(h^2 v_n^1) + o(1) \\
&\le \lambda_2 \int \frac{h^2|v_n^1|^{2^*(s_2)}}{|x|^{s_2}} + \int \nabla g_n \cdot \nabla(h^2 v_n^1) + o(1) \\
&\le \lambda_2 \mu_{s_2}(\mathbb{R}^N)^{-1} \Big(\int_{B(0,1)} \frac{|u_n^1|^{2^*(s_2)}}{|x|^{s_2}} \Big)^{\frac{2-2^*(s_2)}{2}} \int |\nabla(hv_n^1)|^2 + o(1) \\
&\le \frac{1}{2} \int |\nabla(hv_n^1)|^2 + o(1).
\end{aligned}$$

Hence, $\nabla v_n^1 \to 0$ in $L^2_{loc}\big(B(0,1)\big)$ and $v_n^1 \to 0$ in $L^{2^*(s_2)}_{loc}\big(B(0,1), |x|^{-s_2}dx\big)$, which contradicts the fact that

$$\int_{B(0,1)} \Big(\lambda_2 \frac{|v_n^1|^{2^*(s_2)}}{|x|^{s_2}}\Big) dx = \delta > 0.$$

Thus $U^1 \not\equiv 0$ is also true for the case of $\lambda_1 < 0$. In either case, we will prove that $r_n^1 \to 0$. If not, since Ω is bounded, we may assume that $r_n^1 \to r_\infty^1 > 0$, the fact that $u_n^1 \rightharpoonup 0$ in $H_0^1(\Omega)$ means that $v_n^1(x) := (r_n^1)^{\frac{N-2}{2}} u_n^1(r_n^1 x) \rightharpoonup 0$ in $D_0^{1,2}(\mathbb{R}^N)$, which contradicts the fact $U^1 \not\equiv 0$, and therefore $r_n^1 \to 0$.

Next, we prove that $supp\ U^1 \subset \mathbb{R}^N_+$. Without loss of generality, assume that $\partial R^N_+ := \{x_N = 0\}$ is tangent to $\partial\Omega$ at 0, and that $-e_N = (0, \cdots, -1)$ is the outward normal to $\partial\Omega$ at that point. For any compact $K \subset \mathbb{R}^N_-$, we have for n large enough, that $\frac{\Omega}{r_n^1} \cap K = \emptyset$ as $r_n^1 \to 0$. Since $supp\ v_n^1 \subset \frac{\Omega}{r_n^1}$ and $v_n^1 \to U^1$ a.e. in $\mathbb{R}^N$, it follows that $U^1 = 0$ a.e. on K, and therefore $supp\ U^1 \subset \mathbb{R}^N_+$. By (3.10) and Lemma 3.1, $A'_{\lambda_1,\lambda_2}(U^1) = 0$ and U^1 is a weak solution of (3.3). The sequence $u_n^2(x) := u_n^1(x) - (r_n^1)^{\frac{2-N}{2}} U^1(\frac{x}{r_n^1})$ also satisfies

$$\begin{cases} \|u_n^2\|^2 = \|u_n\|^2 - \|U^0\|^2 - \|U^1\|^2 + o(1), \\ A_{\lambda_1,\lambda_2}(u_n^2) \to c - \Phi(U^0) - A_{\lambda_1,\lambda_2}(U^1), \\ A'_{\lambda_1,\lambda_2}(u_n^2) \to 0 \text{ in } H^{-1}(\Omega). \end{cases} \tag{3.14}$$

Moreover,

$$A_{\lambda_1,\lambda_2}(U^1) \geq c_{\lambda_1,\lambda_2} > 0.$$

By iterating the above procedure, we construct similarly sequences $U^j, (r_n^j)$ with the above properties and U^j is a solution of (3.3). It is easy to see that the iteration must terminate after a finite number of steps. □

Next, we will prove that $c_0 := \inf_{u\in\mathcal{N}} \Phi(u) < c_{\lambda_1,\lambda_2}$. Firstly, we recall the following result.

Lemma 3.3. *([11, Theorem 1.1]) Suppose Ω is a bounded smooth domain in $\mathbb{R}^N$, $0 \in \partial\Omega$ and the mean curvature $H(0) < 0$. Then the equation*

$$\begin{cases} \Delta u + \lambda \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 & \text{in } \Omega, \\ u(x) > 0 \quad \text{in } \Omega, \quad u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

has a least-energy solution if $N \geq 3, \lambda \in \mathbb{R}$ and $0 < s_2 < s_1 < 2$.

Remark 3.1. *Let c_1 be the least energy corresponding to (3.15). It has been proved that $c_1 < c_{\lambda,1}$. We refer to [11, Lemma 4.1].*

Corollary 3.2. *Suppose Ω is a bounded smooth domain in $\mathbb{R}^N$, $0 \in \partial\Omega$ and the mean curvature $H(0) < 0$. Assume $N \geq 3, 0 < s_2 < s_1 < 2, 0 \leq s_3 < 2, \lambda_2, \lambda_3 > 0, 1 < p < 2^*(s_3) - 1$. Furthermore, $\lambda_1 > 0$ or $\lambda_1 < 0$ with $p \geq 2^*(s_1) - 1$, then*

$$c_0 := \inf_{u\in\mathcal{N}} \Phi(u) < c_{\lambda_1,\lambda_2}. \tag{3.16}$$

Proof. It is easy to see that

$$\begin{cases} \Delta u + \lambda_1 \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \lambda_2 \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 & \text{in } \Omega, \\ u(x) > 0 \quad \text{in } \Omega \quad u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.17}$$

has a least-energy solution for $\lambda_2 > 0, N \geq 3, \lambda_1 \in \mathbb{R}, 0 < s_2 < s_1 < 2$ (thanks to Lemma 3.3). For this case, we denote the corresponding least energy by $\hat{c}_{\lambda_1,\lambda_2}$. Let $\lambda = \lambda_1 \lambda_2^{\frac{2-2^*(s_1)}{2^*(s_2)-2}}$, by (3.5) and Remark 3.1, we have

$$\hat{c}_{\lambda_1,\lambda_2} = \lambda_2^{\frac{-2}{2^*(s_2)-2}} c_1 < \lambda_2^{\frac{-2}{2^*(s_2)-2}} c_{\lambda,1} = c_{\lambda_1,\lambda_2}. \tag{3.18}$$

We note that the assumptions required in Lemma 2.1 are satisfied. Let $w \in H_0^1(\Omega)$ be a least-energy solution to (3.17). It is easy to see that $J(w) = \max_{t>0} J(tw) = \hat{c}_{\lambda_1,\lambda_2}$, where

$$J(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \frac{\lambda_1}{2^*(s_1)} \int_\Omega \frac{|w|^{2^*(s_1)}}{|x|^{s_1}} dx - \frac{\lambda_2}{2^*(s_2)} \int_\Omega \frac{|w|^{2^*(s_2)}}{|x|^{s_2}} dx.$$

Then for such a w, there exists some $t_w > 0$ such that $t_w w \in \mathcal{N}$. It follows that

$$c_0 := \inf_{u\in\mathcal{N}} \Phi(u) \leq \Phi(t_w w) < J(t_w w) \leq J(w) = \hat{c}_{\lambda_1,\lambda_2} < c_{\lambda_1,\lambda_2}.$$

□

However, for the case of $\lambda_3 < 0$, similar to the arguments of [4, Lemma 3.1], we can prove that $c_0 \geq \hat{c}_{\lambda_1,\lambda_2}$. Nevertheless, the following lemma shows that $c_0 < c_{\lambda_1,\lambda_2}$.

Lemma 3.4. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$. Suppose that $\partial\Omega$ is C^2 at 0 and the mean curvature $H(0) < 0$. We also assume that $N \geq 3, 0 < s_2 < s_1 < 2, 0 \leq s_3 < 2, \lambda_2 > 0, \lambda_3 < 0, 1 < p < 2^*(s_3) - 1$ and $\begin{cases} \lambda_1 > 0 \\ p \leq 2^*(s_1) - 1 \end{cases}$ or $\begin{cases} \lambda_1 < 0 \\ p < 2^*(s_2) - 1 \end{cases}$. Then we have $c_0 := \inf_{u\in\mathcal{N}} \Phi(u) < c_{\lambda_1,\lambda_2}$ if one of the following additional conditions is satisfied:*

(1) $p < \frac{N-2s_3}{N-2}$.

(2) $p \geq \frac{N-2s_3}{N-2}$, $|\lambda_3|$ *is sufficiently small.*

Proof. We prove this lemma by a modification of [9, Lemma 2.2]. Without loss of generality, we may assume that in a neighborhood of 0, $\partial\Omega$ can be represented by $x_N = \varphi(x')$, where $x' = (x_1, \cdots, x_{N-1}), \varphi(0) = 0, \nabla'\varphi(0) = 0, \nabla' = (\partial_1, \cdots, \partial_{N-1})$ and the outer normal of $\partial\Omega$ at 0 is $-e_N = (0, \cdots, 0, -1)$. Define $\phi(x) = (x', x_N - \varphi(x'))$ to "flatten out" the boundary. We can choose a small $r_0 > 0$ and neighborhoods of 0, U and $\tilde{U}$, such that $\phi(U) = B_{r_0}(0),\ \phi(U \cap \Omega) = B^+_{r_0}(0), \phi(\tilde{U}) = B_{\frac{r_0}{2}}(0)$ and $\phi(\tilde{U} \cap \Omega) = B^+_{\frac{r_0}{2}}(0)$. Here we adopt the notation: $B^+_{r_0}(0) = B_{r_0} \cap \mathbb{R}^N_+$ for any $r_0 > 0$. Since $\partial\Omega \in C^2$, φ can be expanded by

$$\varphi(y') = \sum_{i=1}^{N-1} \alpha_i y_i^2 + o(|y'|^2). \tag{3.19}$$

Then

$$H(0) = \frac{1}{N-1} \sum_{i=1}^{N-1} \alpha_i.$$

Suppose that $u \in H^1_0(\mathbb{R}^N_+)$ is a least-energy solution of (3.3), i.e.,

$$\begin{cases} \Delta u + \lambda_1 \frac{u^{2^*(s_1)-1}}{|x|^{s_1}} + \lambda_2 \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 & \text{in } \mathbb{R}^N_+, \\ u(x) > 0 \quad \text{in } \mathbb{R}^N_+, \quad u(x) = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{3.20}$$

and

$$A_{\lambda_1,\lambda_2}(u) = \frac{1}{2} a(u) - \frac{\lambda_1}{2^*(s_1)} b(u) - \frac{\lambda_2}{2^*(s_2)} c(u) = c_{\lambda_1,\lambda_2},$$

where $a(u), b(u), c(u)$ are defined by (2.5). We also note that

$$\max_{t>0} A_{\lambda_1,\lambda_2}(tu) = A_{\lambda_1,\lambda_2}(u) = c_{\lambda_1,\lambda_2}. \tag{3.21}$$

Let $\varepsilon > 0$, we define

$$v_\varepsilon(x) := \varepsilon^{-\frac{N-2}{2}} u(\frac{\phi(x)}{\varepsilon}) \text{ for } x \in \Omega \cap U.$$

Let $\eta \in C_0^\infty(U)$ be a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$ and consider $\hat{v}_\varepsilon := \eta v_\varepsilon$ in Ω, then for $t \geq 0$, if $\lambda_1 > 0$, we have

$$\begin{aligned}
\Phi(t\hat{v}_\varepsilon) =& \frac{t^2}{2}\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx - \lambda_1 \frac{t^{2^*(s_1)}}{2^*(s_1)}\int_\Omega \frac{\hat{v}_\varepsilon^{2^*(s_1)}}{|x|^{s_1}}dx \\
&- \lambda_2 \frac{t^{2^*(s_2)}}{2^*(s_2)}\int_\Omega \frac{\hat{v}_\varepsilon^{2^*(s_2)}}{|x|^{s_2}}dx - \lambda_3 \frac{t^{p+1}}{p+1}\int_\Omega \frac{\hat{v}_\varepsilon^{p+1}}{|x|^{s_3}}dx \\
\leq& \frac{t^2}{2}\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx - \lambda_1 \frac{t^{2^*(s_1)}}{2^*(s_1)}\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s_1)}}{|x|^{s_1}}dx \\
&- \lambda_2 \frac{t^{2^*(s_2)}}{2^*(s_2)}\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s_2)}}{|x|^{s_2}}dx - \lambda_3 \frac{t^{p+1}}{p+1}\int_\Omega \frac{\hat{v}_\varepsilon^{p+1}}{|x|^{s_3}}dx.
\end{aligned}$$

If $\lambda_1 < 0$, we have

$$\begin{aligned}
\Phi(t\hat{v}_\varepsilon) \leq& \frac{t^2}{2}\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx - \lambda_1 \frac{t^{2^*(s_1)}}{2^*(s_1)}\int_{\Omega\cap U} \frac{v_\varepsilon^{2^*(s_1)}}{|x|^{s_1}}dx \\
&- \lambda_2 \frac{t^{2^*(s_2)}}{2^*(s_2)}\int_{\Omega\cap\tilde{U}} \frac{v_\varepsilon^{2^*(s_2)}}{|x|^{s_2}}dx - \lambda_3 \frac{t^{p+1}}{p+1}\int_\Omega \frac{\hat{v}_\varepsilon^{p+1}}{|x|^{s_3}}dx.
\end{aligned}$$

By the change of the variable $y = \frac{\phi(x)}{\varepsilon}$, we have

$$\begin{aligned}
\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx =& \int_{\Omega\cap U} \eta^2 |\nabla v_\varepsilon|^2 dx - \int_{\Omega\cap U} \eta(\Delta\eta) v_\varepsilon^2 dx \\
\leq& \int_{\mathbb{R}^N_+} |\nabla u(y)|^2 dy - 2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y)\nabla' u(y)\cdot(\nabla'\varphi)(\varepsilon y')dy \\
&+ \int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \big(\partial_N u(y)\big)^2 |(\nabla' u)(\varepsilon y')|^2 dy \\
&- \varepsilon^2 \int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)(\Delta\eta)\big(\phi^{-1}(\varepsilon y)\big) u(y)^2 dy.
\end{aligned}$$

Note that, by using $|\nabla'\phi(y')| = O(|y'|)$ and the decay estimate of $|\nabla u|$ in (3.7), we see that

$$\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \big(\partial_N u(y)\big)^2 |(\nabla' u)(\varepsilon y')|^2 dy \leq C\varepsilon^2 \int_{\mathbb{R}^N} \big(1+|y|\big)^{-2N} |y|^2 dy = O(\varepsilon^2).$$

Hence,

$$\int_\Omega |\nabla \hat{v}_\varepsilon|^2 dx = a(u) - 2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y)\nabla' u(y)\cdot(\nabla'\varphi)(\varepsilon y')dy + O(\varepsilon^2).$$

Using integration by parts and the formulas (3.7), (3.19), we see that

$$
\begin{aligned}
I := & -2\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y) \nabla' u(y) \cdot (\nabla'\varphi)(\varepsilon y') dy \\
= & -\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y) \nabla' u(y) \cdot \nabla'[\varphi(\varepsilon y')] dy \\
= & -\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}} \cap \partial\mathbb{R}^N_+} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y) \nabla' u(y) \varphi(\varepsilon y') dS_y \\
& + \frac{4}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big) \nabla'[\eta\big(\phi^{-1}(\varepsilon y)\big)] \partial_N u(y) \nabla' u(y) \cdot \varphi(\varepsilon y') dy \\
& + \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \nabla' \partial_N u(y) \nabla' u(y) \cdot \varphi(\varepsilon y') dy \\
& + \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y) \sum_{i=1}^{N-1} \partial_{ii} u(y) \varphi(\varepsilon y') dy \\
= & \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y) \sum_{i=1}^{N-1} \partial_{ii} u(y) \varphi(\varepsilon y') dy + O(\varepsilon^2).
\end{aligned}
$$

Applying (3.20) and integration by parts, we obtain that

$$\begin{aligned}
I' :=& \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y) \sum_{i=1}^{N-1} \partial_{ii} u(y)\varphi(\varepsilon y')dy \\
=& \frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y)[\Delta u(y) - \partial_{NN} u(y)]\varphi(\varepsilon y')dy \\
=& -\frac{2}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N u(y)[\lambda_1 \frac{u^{2^*(s_1)-1}}{|y|^{s_1}} + \lambda_2 \frac{u^{2^*(s_2)-1}}{|y|^{s_2}}]\varphi(\varepsilon y')dy \\
& - \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \partial_N[\big(\partial_N u(y)\big)^2]\varphi(\varepsilon y')dy \\
=& -\frac{2}{\varepsilon}\lambda_1 \frac{1}{2^*(s_1)}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{\partial_N[u(y)^{2^*(s_1)}]}{|y|^{s_1}}\varphi(\varepsilon y')dy \\
& - \frac{2}{\varepsilon}\lambda_2 \frac{1}{2^*(s_2)}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{\partial_N[u(y)^{2^*(s_2)}]}{|y|^{s_2}}\varphi(\varepsilon y')dy \\
& + \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}} \cap \partial\mathbb{R}^N_+} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \big(\partial_N u(y)\big)^2 \varphi(\varepsilon y')dS_y + O(\varepsilon^2) \\
=& -\frac{2}{\varepsilon}\lambda_1 \frac{s_1}{2^*(s_1)}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{u(y)^{2^*(s_1)} y_N}{|y|^{s_1+2}}\varphi(\varepsilon y')dy \\
& - \frac{2}{\varepsilon}\lambda_2 \frac{s_2}{2^*(s_2)}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{u(y)^{2^*(s_2)} y_N}{|y|^{s_2+2}}\varphi(\varepsilon y')dy \\
& + \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}} \cap \partial\mathbb{R}^N_+} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \big(\partial_N u(y)\big)^2 \varphi(\varepsilon y')dS_y + O(\varepsilon^2) \\
=:& J_1 + J_2 + J_3 + O(\varepsilon^2).
\end{aligned}$$

Among them

$$\begin{aligned}
J_1 :=& -\frac{2}{\varepsilon}\lambda_1 \frac{s_1}{2^*(s_1)}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{u(y)^{2^*(s_1)} y_N}{|y|^{s_1+2}}\varphi(\varepsilon y')dy \\
=& -\frac{2}{\varepsilon}\lambda_1 \frac{s_1}{2^*(s_1)}\int_{B^+_{\frac{r_0}{\varepsilon}} \setminus B^+_{\frac{r_0/2}{\varepsilon}}} \eta\big(\phi^{-1}(\varepsilon y)\big)^2 \frac{u(y)^{2^*(s_1)} y_N}{|y|^{s_1+2}}\varphi(\varepsilon y')dy \\
& - \frac{2}{\varepsilon}\lambda_1 \frac{s_1}{2^*(s_1)}\int_{B^+_{\frac{r_0/2}{\varepsilon}}} \frac{u(y)^{2^*(s_1)} y_N}{|y|^{s_1+2}}\varphi(\varepsilon y')dy \\
=:& J_{1,1} + J_{1,2},
\end{aligned}$$

and

$$|J_{1,1}|\le C\varepsilon\int_{\{r_0/2\le|\varepsilon y|<r_0\}}|y|^{2^*(s_1)(1-N)+1-s_1}dy=O(\varepsilon^{\frac{N(N-s_1)}{N-2}}).$$

Notice that

$$\varepsilon\int_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0/2}{\varepsilon}}}u(y)^{2^*(s_1)}|y|^{1-s_1}dy=O(\varepsilon^{\frac{N(N-s_1)}{N-2}}).\tag{3.22}$$

By (3.19), (3.22) and using the fact of $u(y',y_N)=u(|y'|,y_N)$, we obtain

$$\begin{aligned}
&J_{1,2}\\
&=-2\varepsilon\lambda_1\frac{s_1}{2^*(s_1)}\sum_{i=1}^{N-1}\alpha_i\int_{\mathbb{R}^N_+}\frac{u(y)^{2^*(s_1)}y_N}{|y|^{s_1+2}}y_i^2(1+o(1))dy+O(\varepsilon^{\frac{N(N-s_1)}{N-2}})\\
&=-\frac{2\lambda_1 s_1\varepsilon}{2^*(s_1)(N-1)}\int_{\mathbb{R}^N_+}\frac{u(y)^{2^*(s_1)}y_N}{|y|^{s_1+2}}|y'|^2dy\Big(\sum_{i=1}^{N-1}\alpha_i\Big)(1+o(1))+O(\varepsilon^{\frac{N(N-s_1)}{N-2}}).
\end{aligned}$$

Thus,

$$J_1=-\frac{2\lambda_1 s_1}{2^*(s_1)}K_1H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2),$$

where

$$K_1:=\int_{\mathbb{R}^N_+}\frac{u(y)^{2^*(s_1)}y_N}{|y|^{s_1+2}}|y'|^2dy.$$

Similarly, we can prove that

$$J_2=-\frac{2\lambda_2 s_2}{2^*(s_2)}K_2H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2),$$

where

$$K_2:=\int_{\mathbb{R}^N_+}\frac{u(y)^{2^*(s_2)}y_N}{|y|^{s_2+2}}|y'|^2dy.$$

Next, we see that

$$\begin{aligned}
J_3=&\frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}\cap\partial\mathbb{R}^N_+}\eta\big(\phi^{-1}(\varepsilon y)\big)^2\big(\partial_N u(y)\big)^2\varphi(\varepsilon y')dS_y\\
=&\frac{1}{\varepsilon}\int_{\big(B^+_{\frac{r_0}{\varepsilon}}\setminus B^+_{\frac{r_0/2}{\varepsilon}}\big)\cap\partial\mathbb{R}^N_+}\eta\big(\phi^{-1}(\varepsilon y)\big)^2\big(\partial_N u(y)\big)^2\varphi(\varepsilon y')dS_y\\
&+\frac{1}{\varepsilon}\int_{B^+_{\frac{r_0/2}{\varepsilon}}\cap\partial\mathbb{R}^N_+}\eta\big(\phi^{-1}(\varepsilon y)\big)^2\big(\partial_N u(y)\big)^2\varphi(\varepsilon y')dS_y\\
=:&J_{3,1}+J_{3,2},
\end{aligned}$$

By the mean value theorem for integrals, we have

$$\begin{aligned}
|J_{3,1}|\le&\frac{C(r_0)}{\varepsilon}\big(\frac{1}{\varepsilon}\big)^{-2N}\big(\frac{1}{\varepsilon}\big)^{N-1}\\
=&O(\varepsilon^{2N-1-(N-1)})=O(\varepsilon^N).
\end{aligned}$$

Using the symmetry, by the polar coordinates transformation, we also obtain that

$$\varepsilon\int_{\{|\varepsilon y'|>\frac{r_0}{2}\}}|(\partial_N u)(y',0)|^2|y'|^2dy'=O(\varepsilon^N). \tag{3.23}$$

Thus, by (3.19), (3.23) and using the fact of $u(y',y_N)=u(|y'|,y_N)$, we obtain

$$\begin{aligned}J_{3,2}=&\varepsilon\sum_{i=1}^{N-1}\alpha_i\int_{\mathbb{R}^{N-1}}\big((\partial_N u)(y',0)\big)^2y_i^2dy'\big(1+o(1)\big)+O(\varepsilon^N)\\=&\frac{\varepsilon}{N-1}\int_{\mathbb{R}^{N-1}}\big((\partial_N u)(y',0)\big)^2|y'|^2dy'\sum_{i=1}^{N-1}\alpha_i+O(\varepsilon^2)\\=&K_3H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2),\end{aligned}$$

where

$$K_3:=\int_{\mathbb{R}^{N-1}}\big((\partial_N u)(y',0)\big)^2|y'|^2dy'>0.$$

Hence,

$$I'=\big(K_3-\frac{2\lambda_1s_1}{2^*(s_1)}K_1-\frac{2\lambda_2s_2}{2^*(s_2)}K_2\big)H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2),$$

which implies that

$$I=\big(K_3-\frac{2\lambda_1s_1}{2^*(s_1)}K_1-\frac{2\lambda_2s_2}{2^*(s_2)}K_2\big)H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2)$$

and that

$$\int_\Omega|\nabla\hat{v}_\varepsilon|^2dx=a(u)+\big(K_3-\frac{2\lambda_1s_1}{2^*(s_1)}K_1-\frac{2\lambda_2s_2}{2^*(s_2)}K_2\big)H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2).$$

Furthermore, the integrals $\int_{\Omega\cap\tilde{U}}\frac{v_\varepsilon^{2^*(s_2)}}{|x|^{s_2}}dx$, $\int_{\Omega\cap\tilde{U}}\frac{v_\varepsilon^{2^*(s_1)}}{|x|^{s_1}}dx$ and $\int_{\Omega\cap U}\frac{v_\varepsilon^{2^*(s_1)}}{|x|^{s_1}}dx$ can be estimated by the same argument as that in [9, Lemma 2.2] to obtain that

$$\int_{\Omega\cap\tilde{U}}\frac{v_\varepsilon^{2^*(s_2)}}{|x|^{s_2}}dx=c(u)-s_2K_2H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2)$$

and

$$\int_\Omega\frac{(\hat{v}_\varepsilon)^{2^*(s_1)}}{|x|^{s_1}}dx=b(u)-s_1K_1H(0)\big(1+o(1)\big)\varepsilon+O(\varepsilon^2).$$

By [4, Lemma 2.4], we also obtain that

$$\int_\Omega\frac{(\hat{v}_\varepsilon)^{p+1}}{|x|^{s_3}}dx=\varepsilon^{s_0-s_3}\int_{\mathbb{R}^N_+}\frac{u^{p+1}}{|y|^{s_3}}dy\big(1+o(1)\big)+O(\varepsilon^{\frac{N(p+1)}{2}}),$$

where $s_0 := \frac{N+2}{2} - \frac{N-2}{2}p \in (s_3, 2)$ when $1 < p < 2^*(s_3) - 1$. Thus, we have

$$\begin{aligned}
\Phi(t\hat{v}_\varepsilon) \leq & \frac{t^2}{2}\big[a(u) + \big(K_3 - \frac{2\lambda_1 s_1}{2^*(s_1)}K_1 - \frac{2\lambda_2 s_2}{2^*(s_2)}K_2\big)H(0)\big(1+o(1)\big)\varepsilon + O(\varepsilon^2)\big] \\
& - \lambda_1 \frac{t^{2^*(s_1)}}{2^*(s_1)}\big[b(u) - s_1 K_1 H(0)\big(1+o(1)\big)\varepsilon + O(\varepsilon^2)\big] \\
& - \lambda_2 \frac{t^{2^*(s_2)}}{2^*(s_2)}\big[c(u) - s_2 K_2 H(0)\big(1+o(1)\big)\varepsilon + O(\varepsilon^2)\big] \\
& - \lambda_3 \frac{t^{p+1}}{p+1}\big[\int_{\mathbb{R}^N_+} \frac{u^{p+1}}{|y|^{s_3}}dy\big(1+o(1)\big)\varepsilon^{s_0-s_3} + O(\varepsilon^{\frac{N(p+1)}{2}})\big].
\end{aligned}$$

Then, it is easy to see that there exists some T large enough and ε_0 sufficiently small such that $\Phi(T\hat{v}_\varepsilon) < 0$ for all $\varepsilon < \varepsilon_0$. By Lemma 2.1 and Remark 2.1, there exists a unique $t_{\hat{v}_\varepsilon} > 0$ such that $t_{\hat{v}_\varepsilon}\hat{v}_\varepsilon \in \mathcal{N}$ and $\max_{t>0}\Phi(t\hat{v}_\varepsilon) = \Phi(t_{\hat{v}_\varepsilon}\hat{v}_\varepsilon) \geq c_0 > 0$. Hence, we obtain that

$$t_{\hat{v}_\varepsilon} < T \text{ for all } \varepsilon < \varepsilon_0. \tag{3.24}$$

On the other hand, when $t \in (0, T)$, we have

$$\begin{aligned}
\Phi(t\hat{v}_\varepsilon) =& A_{\lambda_1,\lambda_2}(tu) + \big[\frac{t^2}{2}\big(K_3 - \frac{2\lambda_1 s_1}{2^*(s_1)}K_1 - \frac{2\lambda_2 s_2}{2^*(s_2)}K_2\big) \\
& + \lambda_1 s_1 K_1 \frac{t^{2^*(s_1)}}{2^*(s_1)} + \lambda_2 s_2 K_2 \frac{t^{2^*(s_2)}}{2^*(s_2)}\big] H(0)\big(1+o(1)\big)\varepsilon \\
& - \lambda_3 \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N_+} \frac{u^{p+1}}{|y|^{s_3}}dy\big(1+o(1)\big)\varepsilon^{s_0-s_3} + O(\varepsilon^2) \\
=:& A_{\lambda_1,\lambda_2}(tu) + g_1(t)H(0)\big(1+o(1)\big)\varepsilon \\
& - \lambda_3 \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N_+} \frac{u^{p+1}}{|y|^{s_3}}dy\big(1+o(1)\big)\varepsilon^{s_0-s_3} + O(\varepsilon^2).
\end{aligned}$$

Notice that $g_1(1) = \frac{1}{2}K_3 > 0$, there exists $\delta_0 > 0, \varepsilon_1 < \varepsilon_0$ such that

$$g_1(t) \geq \frac{1}{4}K_3 > 0 \text{ for all } (t, \varepsilon) \in [1-\delta_0, 1+\delta_0] \times (0, \varepsilon_1).$$

Define

$$M := \max_{t \in (0, 1-\delta_0] \cup [1+\delta_0, T]} A_{\lambda_1,\lambda_2}(tu),$$

we see that $M < c_{\lambda_1,\lambda_2}$ by (3.21). Then, by the continuity, we obtain that for $\varepsilon < \varepsilon_2$ small enough,

$$\max_{t \in (0, 1-\delta_0] \cup [1+\delta_0, T]} \Phi(t\hat{v}_\varepsilon) \leq M + O(\varepsilon^\sigma) < c_{\lambda_1,\lambda_2}, \tag{3.25}$$

where $\sigma := \min\{s_0 - s_3, 1\} > 0$. On the other hand, recalling that $H(0) < 0, g_1(t) > \frac{1}{4}K_3$ for all $1 - \delta_0 \leq t \leq 1 + \delta_0$. When $s_0 - s_3 > 1$, that is, $p < \frac{N-2s_3}{N-2}$, then it is

easy to see that there exists $\varepsilon_3 < \varepsilon_2$ such that

$$g_1(t)H(0)\big(1+o(1)\big)\varepsilon - \lambda_3 \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N_+} \frac{u^{p+1}}{|y|^{s_3}}dy\big(1+o(1)\big)\varepsilon^{s_0-s_3} + O(\varepsilon^2) < 0$$

for all $(t,\varepsilon) \in [1-\delta_0, 1+\delta_0] \times (0,\varepsilon_3)$. It follows that

$$\max_{1-\delta_0\le t\le 1+\delta_0} \Phi(t\hat{v}_\varepsilon) < \max_{1-\delta_0\le t\le 1+\delta_0} A_{\lambda_1,\lambda_2}(tu) = A_{\lambda_1,\lambda_2}(u) = c_{\lambda_1,\lambda_2}. \tag{3.26}$$

Combine with (3.24), (3.25) and (3.26), we obtain that

$$c_0 \le \max_{t>0} \Phi(t\hat{v}_\varepsilon) < c_{\lambda_1,\lambda_2}$$

for ε small enough. Similar to Corollary 3.2, let $w \in H_0^1(\Omega)$ be a least-energy solution to (3.17). Then

$$J(w) = \max_{t>0} J(tw) = \hat{c}_{\lambda_1,\lambda_2} < c_{\lambda_1,\lambda_2},$$

where

$$J(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \frac{\lambda_1}{2^*(s_1)}\int_\Omega \frac{|w|^{2^*(s_1)}}{|x|^{s_1}}dx - \frac{\lambda_2}{2^*(s_2)}\int_\Omega \frac{|w|^{2^*(s_2)}}{|x|^{s_2}}dx.$$

Since $2^*(s_2) > 2^*(s_1)$ and $\lambda_2 > 0$, it is easy to see that there exists some $T > 0$ such that $J(Tw) < 0$. When $|\lambda_3|$ is small enough, we have $\Phi(Tw) < 0$. By Lemma 2.1 again, there exists some $0 < t_w < T$ such that $t_w w \in \mathcal{N}$ and

$$\max_{t>0} \Phi(tw) = \Phi(t_w w) \ge c_0 > 0.$$

On the other hand, since $\hat{c}_{\lambda_1,\lambda_2} < c_{\lambda_1,\lambda_2}$, when $|\lambda_3|$ is small enough for the case of $p \ge \frac{N-2s_3}{N-2}$, we have

$$\begin{aligned}
\Phi(t_w w) =& J(t_w w) - \lambda_3 \frac{t_w^{p+1}}{p+1}\int_\Omega \frac{w^{p+1}}{|x|^{s_3}}dx \\
\le& J(w) + |\lambda_3||T|^{p+1}\int_\Omega \frac{w^{p+1}}{|x|^{s_3}}dx \\
=& \hat{c}_{\lambda_1,\lambda_2} + |\lambda_3||T|^{p+1}\int_\Omega \frac{w^{p+1}}{|x|^{s_3}}dx \\
<& c_{\lambda_1,\lambda_2}.
\end{aligned}$$

□

Proof of Theorem 1.1. Under the assumptions of Theorem 1.1, the Nehari manifold is well defined due to Lemma 2.1. By Ekeland's variational principle, let $\{u_n\} \subset \mathcal{N}$ be a minimizing sequence such that $\Phi(u_n) \to c_0 := \inf_{u\in\mathcal{N}} \Phi(u)$ and $\Phi'|_{\mathcal{N}}(u_n) \to 0$. By Lemma 2.2 and Lemma 2.3, we obtain that $\{u_n\}$ is a bounded $(PS)_{c_0}$ sequence of Φ. By Corollary 3.2 or Lmema 3.4, we can also obtain that $c_0 < c_{\lambda_1,\lambda_2}$. Hence, by

Corollary 3.1, $\Phi(u)$ satisfies $(PS)_{c_0}$ condition. That is, up to a subsequence, $u_n \to u_0$ strongly in $H_0^1(\Omega)$ for some $u_0 \in H_0^1(\Omega)$ and $\Phi'(u_0) = 0, \Phi(u_0) = c_0$. Thereby, the existence of the ground state solution is established. We also note that $\Phi(u)$ is even, which implies that $|u_0| \in \mathcal{N}$ and $\Phi(|u_0|) = \Phi(u_0) = c_0$. Hence, without loss of generality, we may assume that $u_0 \geq 0$. Finally, apply the similar arguments as [12, Proof of Lemma 2.6(i)], we can obtain the similar regularity property for a nonnegative solution. By the maximum principle, we claim that $u > 0$. $\square$.

4 Proof of Theorem 1.2

In this section, we always assume that $\lambda_3 < 0, p \leq 2^*(s_3) - 1$. Denote λ_3 by λ for the simplicity. To obtain a positive solution, we consider the following modified functional

$$I_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda_1}{2^*(s_1)}\int_\Omega \frac{u_+^{2^*(s_1)}}{|x|^{s_1}}dx$$
$$-\frac{\lambda_2}{2^*(s_2)}\int_\Omega \frac{u_+^{2^*(s_2)}}{|x|^{s_2}}dx - \frac{\lambda}{p+1}\int_\Omega \frac{u_+^{p+1}}{|x|^{s_3}}dx.$$

By (3.18), we have $\hat{c}_{\lambda_1,\lambda_2} < c_{\lambda_1,\lambda_2}$ and it is easy to see that $\hat{c}_{\lambda_1,\lambda_2}$ is the ground state value of I_0 (i.e., $\lambda = 0$) which can be obtained by some $0 < u_0 \in H_0^1(\Omega)$. That is,

$$\Delta u_0 + \lambda_1 \frac{u_0^{2^*(s_1)-1}}{|x|^{s_1}} + \lambda_2 \frac{u_0^{2^*(s_2)-1}}{|x|^{s_2}} = 0$$

in Ω and $I_0(u_0) = \hat{c}_{\lambda_1,\lambda_2}$. Next, for the convenience, we denote $\hat{c}_{\lambda_1,\lambda_2}$ by c_0. It is easy to prove that u_0 is a mountain pass type solution. Here we list the conditions which are fulfilled by I_0 without proof (see [4, Section 5]):

(M1) there exists $c, r > 0$ such that if $\|u\| = r$, then $I_0(u) \geq c$ and there exists $v_0 \in H_0^1(\Omega)$ such that $\|v_0\| > r$ and $I_0(v_0) < 0$;

(M2) there exists a critical point $u_0 \in H_0^1(\Omega)$ of I_0 such that

$$I_0(u_0) = c_0 := \min_{\gamma\in\Gamma}\max_{t\in[0,1]} I_0(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) | \gamma(0) = 0, \gamma(1) = v_0\}$;

(M3) it holds that $c_0 := \inf\{I_0(u) | u \in H_0^1(\Omega)\backslash\{0\}\}$;

(M4) the set $\mathcal{S} := \{u \in H_0^1(\Omega) | I_0'(u) = 0, I_0(u) = c_0\}$ is compact in $H_0^1(\Omega)$;

(M5) there exists a curve $\gamma_0(t) \in \Gamma$ passing through u_0 at $t = t_0$ and satisfying

$$I_0(u_0) > I_0(\gamma_0(t)) \text{ for all } t \neq t_0.$$

Similar to [10, 4], we define a modified mountain pass energy level of I_λ

$$c_\lambda := \min_{\gamma\in\Gamma_M} \max_{0\le t\le 1} I_\lambda(\gamma(t)),$$

where

$$\Gamma_M = \Big\{\gamma\in\Gamma| \sup_{0\le t\le 1}\|\gamma(t)\|\le M\Big\} \text{ with } M := 2\max\Big\{\sup_{u\in\mathcal{S}}\|u\|, \sup_{t\in[0,1]}\|\gamma_0(t)\|\Big\} \text{ fixed.}$$

Then it is easy to see that $\gamma_0 \in \Gamma_M$ and thus $c_0 := \min_{\gamma\in\Gamma_M}\max_{0\le t\le 1} I_0(\gamma(t))$. Similar to [4, 10], we can easily prove the following results although the functional is different. Since the proofs are analogous, we omit the details and refer the readers to [4, Lemma 5.1-5.3, Proposition 5.1-5.3].

Lemma 4.1. *$c_\lambda \ge c_0$ and $\lim_{\lambda\to 0} c_\lambda = c_0$.*

Lemma 4.2. *For $\forall\ d>0$, let $\{u_j\}\subset\mathcal{S}^d$, then up to a subsequence, $u_j \rightharpoonup u\in\mathcal{S}^{2d}$.*

Lemma 4.3. *Suppose that there exist sequences $\lambda_j<0, \lambda_j\to 0$ and $\{u_j\}\subset\mathcal{S}^d$ satisfying $\lim\limits_{j\to\infty} I_{\lambda_j}(u_j)\le c_0$ and $\lim\limits_{j\to\infty} I'_{\lambda_j}(u_j)=0$. Then there is $d_1>0$ such that for $0<d<d_1$, $\{u_j\}$ converges to some $u\in\mathcal{S}$ up to a subsequence.*

Next, we define $m_\lambda := \max\limits_{0\le t\le 1} I_\lambda(\gamma_0(t))$. Then by the definition of c_λ we have that $c_\lambda\le m_\lambda$. It is easy to see that $\lim\limits_{\lambda\to 0} m_\lambda\le c_0$. Combining with the conclusion of Lemma 4.1, we obtain that

$$c_\lambda\le m_\lambda \text{ and } \lim_{\lambda\to 0} c_\lambda = \lim_{\lambda\to 0} m_\lambda = c_0. \tag{4.1}$$

We also define $I_\lambda^{m_\lambda} := \{u\in H_0^1(\Omega)|I_\lambda(u)\le m_\lambda\}$.

Lemma 4.4. *For any $d_2, d_3>0$ satisfying $d_3<d_2<d_1$, there are constant $\alpha>0$ and $\lambda_0<0$ depending on d_2, d_3 such that for $\lambda\in(\lambda_0,0)$, we have*

$$\|I'_\lambda(u)\|\ge\alpha \textit{ for all } u\in I_\lambda^{m_\lambda}\cap(\mathcal{S}^{d_2}\backslash\mathcal{S}^{d_3}).$$

Lemma 4.5. *For $d>0$, there exists $\delta>0$ such that if $\lambda<0$ with $|\lambda|$ small enough,*

$$t\in[0,1], I_\lambda(\gamma_0(t))\ge c_\lambda-\delta \textit{ implies } \gamma_0(t)\in\mathcal{S}^d.$$

Lemma 4.6. *For any $d>0$ small and $\lambda<0$ with $|\lambda|$ small enough depending on d, there exists a sequence $\{u_j\}\subset\mathcal{S}^d\cap I_\lambda^{m_\lambda}$ such that $I'_\lambda(u_j)\to 0$ as $j\to\infty$.*

Proof of Theorem1.2. Taking $d>0$ small enough, by Lemma 4.6, there exists some small $\lambda_0<0$ such that for $\lambda_0<\lambda<0$, there exits a Palais-Smale sequence $\{u_j^\lambda\}\subset\mathcal{S}^{\frac{d}{2}}$. Since $\mathcal{S}$ is compact, it is easy to see $\{u_j^\lambda\}$ is bounded in $H_0^1(\Omega)$. Then by Lemma 4.2, there exists some $u^\lambda\in\mathcal{S}^{2\cdot\frac{d}{2}}=\mathcal{S}^d$ such that $u_j^\lambda\rightharpoonup u^\lambda$ up to a subsequence. Then we obtain $I'_\lambda(u^\lambda)=0$ and $u^\lambda\ne 0$. Hence u^λ is a nontrivial critical point of I_λ. Testing by u_-^λ, we obtain that $\|u_-^\lambda\|^2=0$ which implies $u_-^\lambda\equiv 0$ and $u^\lambda\ge 0$. Finally, by the maximum principle, we know that $u^\lambda>0$. Hence, u^λ is a solution to (1.1). □

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